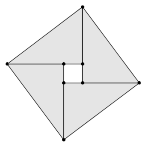
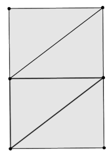
Problems of 2nd Iranian Geometry Olympiad 2015 (Elementary)

1. We have four wooden triangles with sides 3, 4, 5 centimeters. How many convex polygons can we make by all of these triangles?(Just draw the polygons without any proof)
A convex polygon is a polygon which all of it's angles are less than 180° and there isn't any hole in it. For example:



This polygon isn't convex



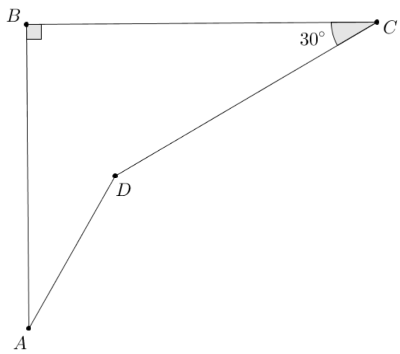
This polygon is convex

Proposed by Mahdi Etesami Fard

2. Let ABC be a triangle with $\angle A = 60°$. The points M, N, K lie on BC, AC, AB respectively such that $BK = KM = MN = NC$. If $AN = 2AK$, find the values of $\angle B$ and $\angle C$.

Proposed by Mahdi Etesami Fard

3. In the figure below, we know that $AB = CD$ and $BC = 2AD$. Prove that $\angle BAD = 30°$.



Proposed by Morteza Saghafian

4. In rectangle $ABCD$, the points M, N, P, Q lie on AB, BC, CD, DA respectively such that the area of triangles AQM, BMN, CNP, DPQ are equal. Prove that the quadrilateral $MNPQ$ is parallelogram.

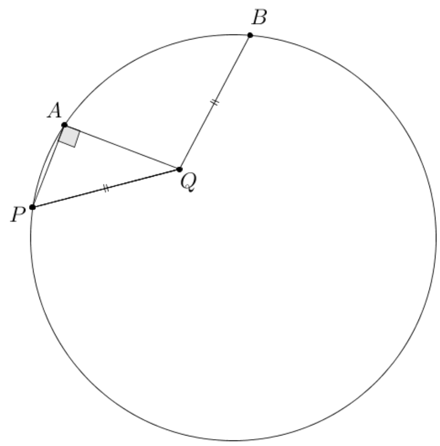
Proposed by Mahdi Etesami Fard

5. Do there exist 6 circles in the plane such that every circle passes through centers of exactly 3 other circles?

Proposed by Morteza Saghafian

Problems of 2nd Iranian Geometry Olympiad 2015 (Medium)

1. In the figure below, the points P, A, B lie on a circle. The point Q lies inside the circle such that $\angle PAQ = 90^\circ$ and $PQ = BQ$. Prove that the value of $\angle AQB - \angle PQA$ is equal to the arc AB.



Proposed by Davood Vakili

2. In acute-angled triangle ABC, BH is the altitude of the vertex B. The points D and E are midpoints of AB and AC respectively. Suppose that F be the reflection of H with respect to ED. Prove that the line BF passes through circumcenter of ABC.

Proposed by Davood Vakili

3. In triangle ABC, the points M, N, K are the midpoints of BC, CA, AB respectively. Let ω_B and ω_C be two semicircles with diameter AC and AB respectively, outside the triangle. Suppose that MK and MN intersect ω_C and ω_B at X and Y respectively. Let the tangents at X and Y to ω_C and ω_B respectively, intersect at Z. prove that $AZ \perp BC$.

Proposed by Mahdi Etesami Fard

4. Let ABC be an equilateral triangle with circumcircle ω and circumcenter O. Let P be the point on the arc BC(the arc which A doesn't lie). Tangent to ω at P intersects extensions of AB and AC at K and L respectively. Show that $\angle KOL > 90°$.

Proposed by Iman Maghsoudi

5. a) Do there exist 5 circles in the plane such that every circle passes through centers of exactly 3 circles?

b) Do there exist 6 circles in the plane such that every circle passes through centers of exactly 3 circles?

Proposed by Morteza Saghafian

Problems of 2nd Iranian Geometry Olympiad 2015 (Advanced)

1. Two circles ω_1 and ω_2 (with centers O_1 and O_2 respectively) intersect at A and B. The point X lies on ω_2. Let point Y be a point on ω_1 such that $\angle XBY = 90°$. Let X' be the second point of intersection of the line O_1X and ω_2 and K be the second point of intersection of $X'Y$ and ω_2. Prove that X is the midpoint of arc AK.

Proposed by Davood Vakili

2. Let ABC be an equilateral triangle with circumcircle ω and circumcenter O. Let P be the point on the arc BC(the arc which A doesn't lie). Tangent to ω at P intersects extensions of AB and AC at K and L respectively. Show that $\angle KOL > 90°$.

Proposed by Iman Maghsoudi

3. Let H be the orthocenter of the triangle ABC. Let l_1 and l_2 be two lines passing through H and perpendicular to each other. l_1 intersects BC and extension of AB at D and Z respectively, and l_2 intersects BC and extension of AC at E and X respectively. Let Y be a point such that $YD \parallel AC$ and $YE \parallel AB$. Prove that X, Y, Z are collinear.

Proposed by Ali Golmakani

4. In triangle ABC, we draw the circle with center A and radius AB. This circle intersects AC at two points. Also we draw the circle with center A and radius AC and this circle intersects AB at two points. Denote these four points by A_1, A_2, A_3, A_4. Find the points B_1, B_2, B_3, B_4 and C_1, C_2, C_3, C_4 similarly. Suppose that these 12 points lie on two circles. Prove that the triangle ABC is isosceles.

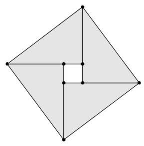
Proposed by Morteza Saghafian

5. Rectangles ABA_1B_2, BCB_1C_2, CAC_1A_2 lie otside triangle ABC. Let C' be a point such that $C'A_1 \perp A_1C_2$ and $C'B_2 \perp B_2C_1$. Points A' and B' are defined similarly. Prove that lines AA', BB', CC' concur.

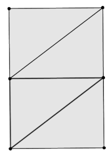
Proposed by Alexey Zaslavsky (Russia)

Solutions of 2nd Iranian Geometry Olympiad 2015 (Elementary)

1. We have four wooden triangles with sides 3, 4, 5 centimeters. How many convex polygons can we make by all of these triangles?(Just draw the polygons without any proof)
A convex polygon is a polygon which all of it's angles are less than 180° and there isn't any hole in it. For example:



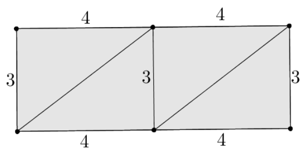
This polygon isn't convex

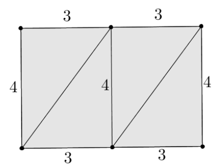


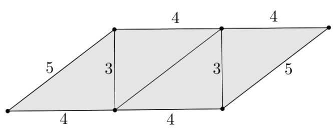
This polygon is convex

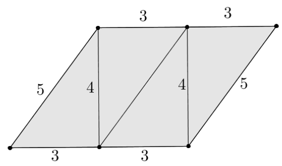
Proposed by Mahdi Etesami Fard

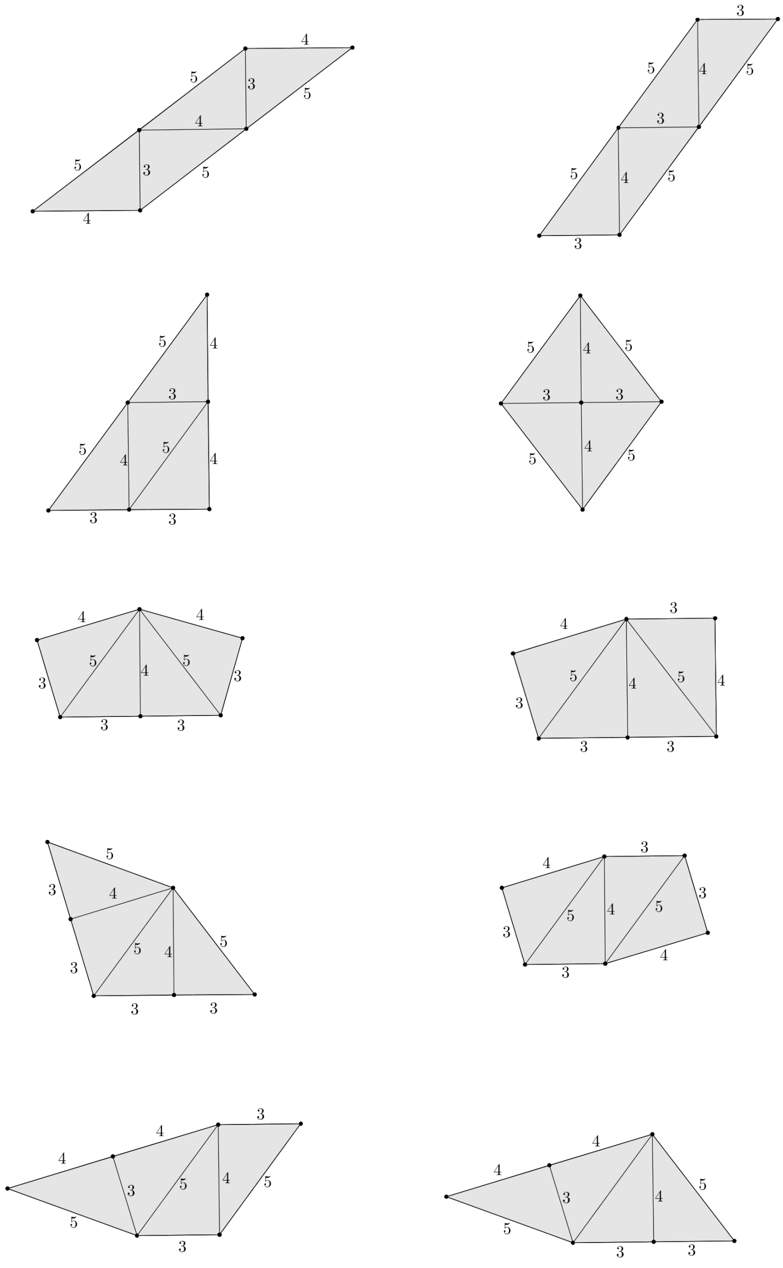
Solution.











2. Let ABC be a triangle with $\angle A = 60°$. The points M, N, K lie on BC, AC, AB respectively such that $BK = KM = MN = NC$. If $AN = 2AK$, find the values of $\angle B$ and $\angle C$.

Proposed by Mahdi Etesami Fard

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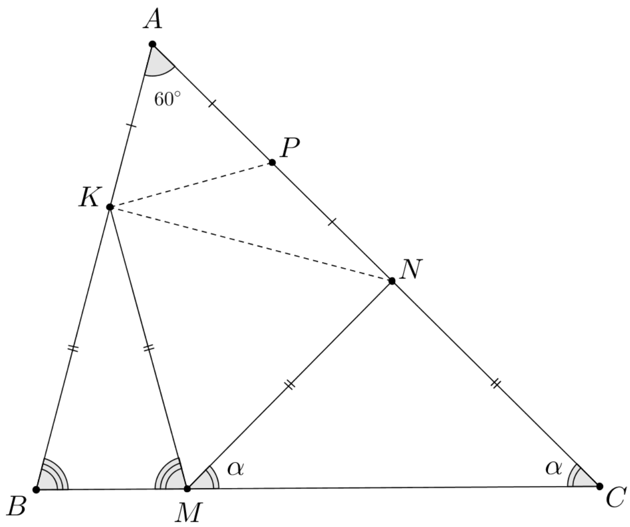
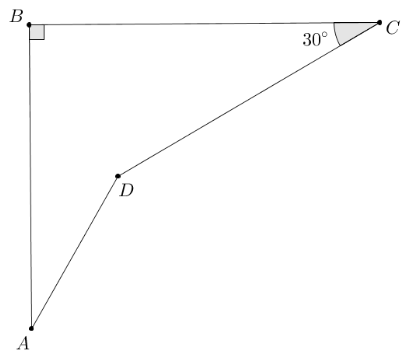
Solution.

Suppose the point P be the midpoint of AN. Therefore $AK = AP = AN$ and so we can say $\triangle APK$ is the equilateral triangle. So $\angle ANK = \frac{\angle KPA}{2} = 30°$ Let $\angle ACB = \angle NMC = \alpha$. Therfore $\angle ABC = \angle KMB = 120° - \alpha$. So $\angle KMN = 60°$. Therefore $\triangle KMN$ is the equilateral triangle. Now we know that $\angle MNA = 90°$. Therefore $\alpha = 45°$. So we have $\angle C = 45°$ and $\angle B = 75°$.



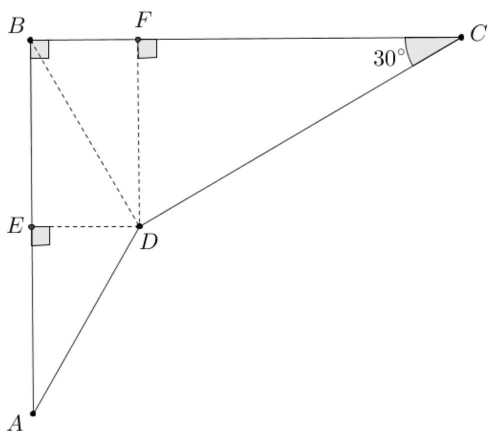
3. In the figure below, we know that $AB = CD$ and $BC = 2AD$. Prove that $\angle BAD = 30°$.



Proposed by Morteza Saghafian

Solution 1.

Let two points E and F on BC and AB respectively such that $DF \perp BC$ and $DE \perp AB$. We can say $DF = \frac{DC}{2} = \frac{AB}{2}$.(because of $\angle BCD = 30°$ and $\angle DFC = 90°$) Also we know that $DF = BE$, therfore DE is the perpendicular bisector of AB. So $BD = AD$.



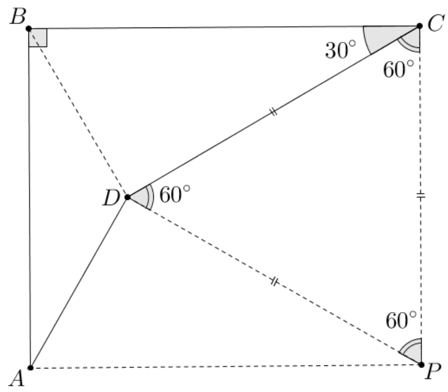
Let H be a point on CD such that $BH \perp CD$. therefore $BH = \frac{BC}{2} = BD$, so we can say $D \equiv H$ and $\angle BDC = 90°$. Therefore $\angle ABD = \angle BAD = 30°$.

Solution 2.

Suppose that P is the point such that triangle DCP is Equilateral. We know that $PC \perp BC$ and $PC = CD = AB$, therfore quadrilateral $ABCP$ is Rectangular.

$$\Rightarrow \quad \angle APD = \angle APC - \angle DPC = 90^\circ - 60^\circ = 30^\circ$$

In other hand, $DP = DC$ and $AP = BC$. So $\triangle ADP$ and $\triangle BDC$ are congruent. Therfore $AD = BD$.



Let the point H on CD such that $BH \perp CD$. therefore $BH = \frac{BC}{2} = BD$, so we can say $D \equiv H$ and $\angle BDC = 90^\circ$. Therefore $\angle ABD = \angle BAD = 30^\circ$.

4. In rectangle $ABCD$, the points M, N, P, Q lie on AB, BC, CD, DA respectively such that the area of triangles AQM, BMN, CNP, DPQ are equal. Prove that the quadrilateral $MNPQ$ is parallelogram.

Proposed by Mahdi Etesami Fard

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Solution.

Let $AB = CD = a, AD = BC = b$ and $AM = x, AQ = z, PC = y, NC = t$. If $x \neq y$, we can assume that $x > y$. We know that:

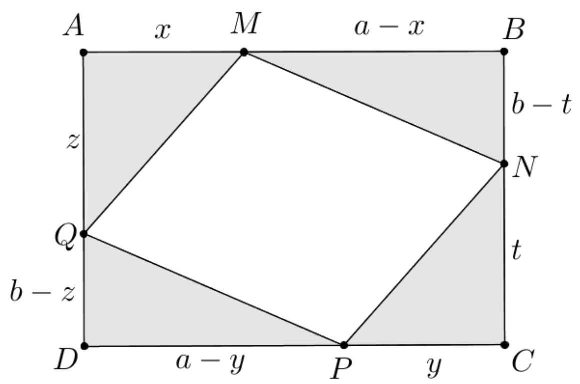
$$y < x \quad \Rightarrow \quad a - x < a - y \qquad (1)$$

$$S_{AQM} = S_{CNP} \quad \Rightarrow \quad zx = yt \quad \Rightarrow \quad z < t \quad \Rightarrow \quad b - t < b - z \qquad (2)$$

According to inequality 1, 2:

$$(a - x)(b - t) < (a - y)(b - z) \quad \Rightarrow \quad S_{BMN} < S_{DPQ}$$

it's a contradiction. Therfore $x = y$, so $z = t$ Now we can say two triangles AMQ and CPN are congruent. Therefore $MQ = NP$ and similarly $MN = PQ$. So the quadrilateral $MNPQ$ is parallelogram.



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Comment.

If quadrilateral $ABCD$ be the parallelogram, similarly we can show that quadrilateral $MNPQ$ is parallelogram.

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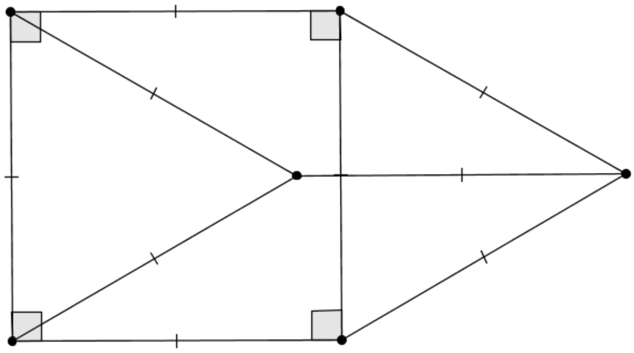
5. Do there exist 6 circles in the plane such that every circle passes through centers of exactly 3 other circles?

Proposed by Morteza Saghafian

Solution.

In the picture below, we have 6 points in the plane such that for every point there exists exactly 3 other points on a circle with radius 1 centimeter.



Solutions of 2nd Iranian Geometry Olympiad 2015 (Medium)

1. In the figure below, the points P, A, B lie on a circle. The point Q lies inside the circle such that $\angle PAQ = 90°$ and $PQ = BQ$. Prove that the value of $\angle AQB - \angle PQA$ is equal to the arc AB.

Proposed by Davood Vakili

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Solution 1.

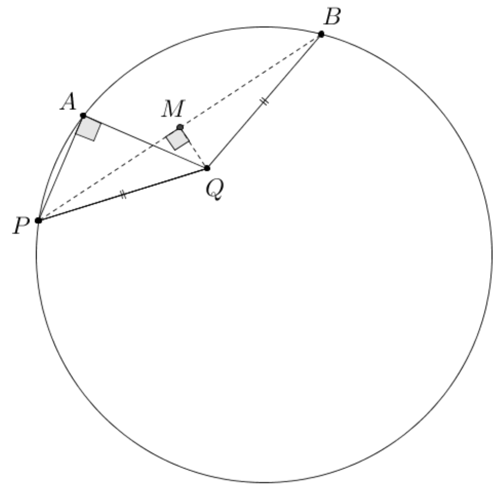
Let point M be the midpoint of PB. So we can say $\angle PMQ = 90°$ and we know that $\angle PAQ = 90°$, therefore quadrilateral $PAMQ$ is cyclic. Therefore:

$$\angle APM = \angle AQM$$

In the other hand:

$$\angle AQB - \angle AQP = \angle PQM + \angle AQM - \angle AQP = 2\angle AQM$$

So we can say that the subtract $\angle AQB$ from $\angle PQA$ is equal to arc AB.



Solution 2.

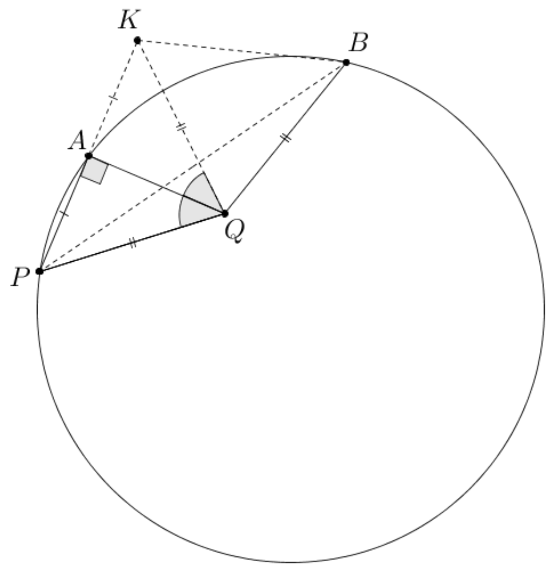
Let the point K be the reflection of P to AQ. We have to show:

$$2\angle APB = \angle AQB - \angle AQP$$

Now we know that AQ is the perpendicular bisector of PK. So $\angle AQP = \angle AQK$ and $PQ = KQ = BQ$, therefore the point Q is the circumcenter of triangle PKB. We know that:

$$2\angle APB = \angle KQB = \angle AQB - \angle AQK = \angle AQB - \angle AQP$$

Therefore the subtract $\angle AQB$ from $\angle PQA$ is equal to arc AB.



2. In acute-angled triangle ABC, BH is the altitude of the vertex B. The points D and E are midpoints of AB and AC respectively. Suppose that F be the reflection of H with respect to ED. Prove that the line BF passes through circumcenter of ABC.

Proposed by Davood Vakili

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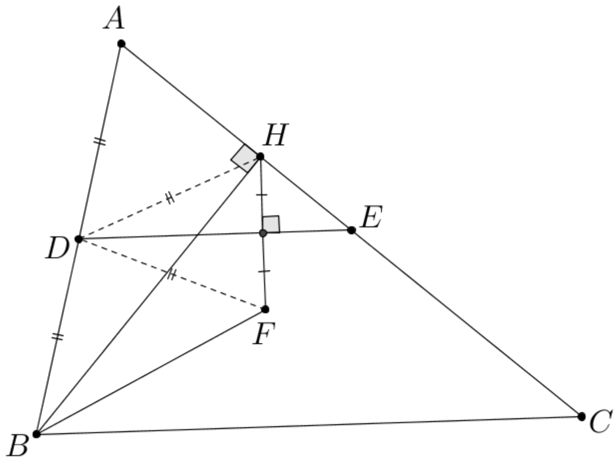
Solution 1.

The circumcenter of $\triangle ABC$ denote by O. We know that $\angle OBA = 90° - \angle C$, therfore we have to show that $\angle FBA = 90° - \angle C$. We know that $AD = BD = DH$, also $DH = DF$.



Therfore quadrilateral $AHFB$ is cyclic (with circumcenter D)

$$\Rightarrow \quad \angle FBA = \angle FHE = 90° - \angle DEH \quad , \quad DE \parallel BC \quad \Rightarrow \quad \angle DEH = \angle C$$

$$\Rightarrow \quad \angle FBA = 90° - \angle C$$

Solution 2.

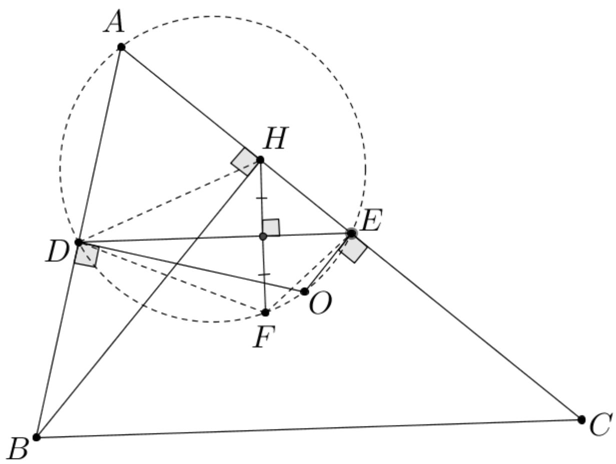
The circumcenter of $\triangle ABC$ denote by O. We know that quadrilateral $ADOE$ is cyclic. Also we know that $AD = HD = DB$, therefore:

$$\angle A = \angle DHA = 180^\circ - \angle DHE = 180^\circ - \angle DFE \quad \Rightarrow \quad ADFE : cyclic$$

So we can say $ADFOE$ is cyclic, therefore quadrilateral $DFOE$ is cyclic.

$$\angle C = \angle DEA = \angle DEF = \angle DOF$$

In the other hand: $\angle C = \angle DOB$ so $\angle DOF = \angle DOB$, therefore B, F, O are collinear.



3. In triangle ABC, the points M, N, K are the midpoints of BC, CA, AB respectively. Let ω_B and ω_C be two semicircles with diameter AC and AB respectively, outside the triangle. Suppose that MK and MN intersect ω_C and ω_B at X and Y respectively. Let the tangents at X and Y to ω_C and ω_B respectively, intersect at Z. prove that $AZ \perp BC$.

Proposed by Mahdi Etesami Fard

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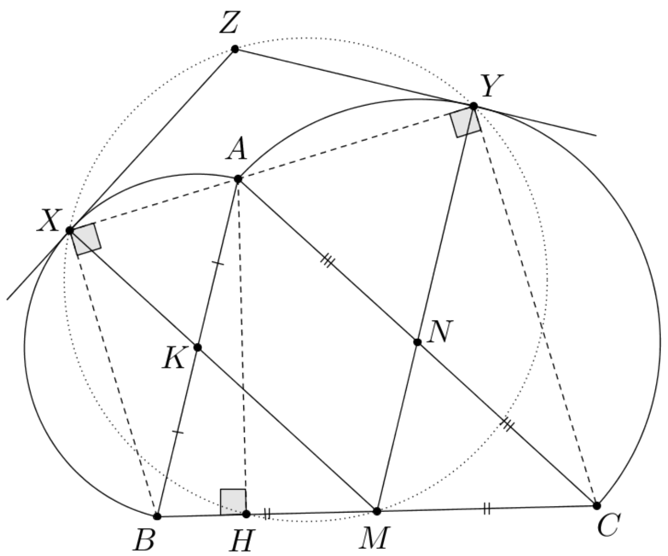
Solution 1.

Let point H on BC such that $AH \perp BC$. Therefore quadrilaterals $AXBH$ and $AYCH$ are cyclic. We know that KM and MN are parallel to AC and AB respectively. So we can say $\angle AKX = \angle ANY = \angle A$, therefore $\angle ABX = \angle ACY = \frac{\angle A}{2}$ and $\angle XAB = \angle YAC = 90° - \frac{\angle A}{2}$. So X, A, Y are collinear.

$$\angle AHX = \angle ABX = \frac{\angle A}{2}, \angle AHY = \angle ACY = \frac{\angle A}{2} \Rightarrow \angle XHY = \angle XMY = \angle A$$



Therefore quadrilateral $XHMY$ is cyclic. Also we know that $\angle MXZ = \angle MYZ = 90°$, therefore quadrilateral $MXZY$ is cyclic. So we can say $ZXHMY$ is cyclic. therfore quadrilateral $HXZY$ is cyclic.

In the other hand: $\angle ZYX = \angle ACY = \frac{\angle A}{2}$

$$\angle ZHX = \angle ZYX = \frac{\angle A}{2}\ ,\ \angle AHX = \frac{\angle A}{2} \quad \Rightarrow \quad \angle ZHX = \angle AHX$$

So the points Z, A, H are collinear, therefore $AZ \perp BC$.

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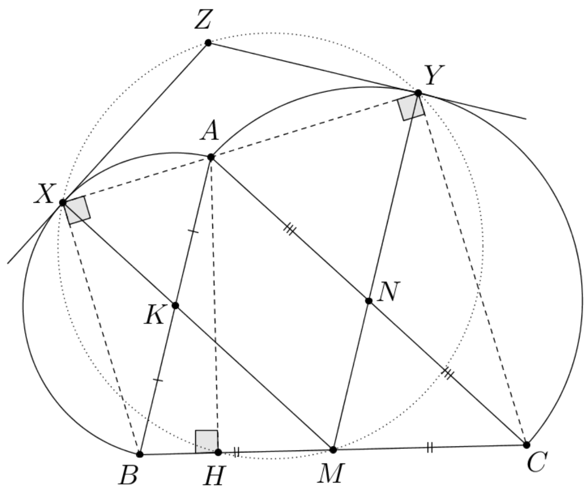
Solution 2.

Let point H on BC such that $AH \perp BC$. We know that KM and MN are parallel to AC and AB respectively. So we can say $\angle AKX = \angle ANY = \angle A$, therefore $\angle ABX = \angle ACY = \frac{\angle A}{2}$ and $\angle XAB = \angle YAC = 90^\circ - \frac{\angle A}{2}$. So X, A, Y are collinear.

$$\Rightarrow \quad \angle ZXY = \angle ZYX = \frac{\angle A}{2} \quad \Rightarrow \quad ZX = ZY$$

So the point Z lie on the radical axis of two these semicirculars. Also we know that the line AH is the radical axis of two these semicirculars. Therefore the points Z, A, H are collinear, therefore $AZ \perp BC$.



4. Let ABC be an equilateral triangle with circumcircle ω and circumcenter O. Let P be the point on the arc BC(the arc which A doesn't lie). Tangent to ω at P intersects extensions of AB and AC at K and L respectively. Show that $\angle KOL > 90°$.

Proposed by Iman Maghsoudi

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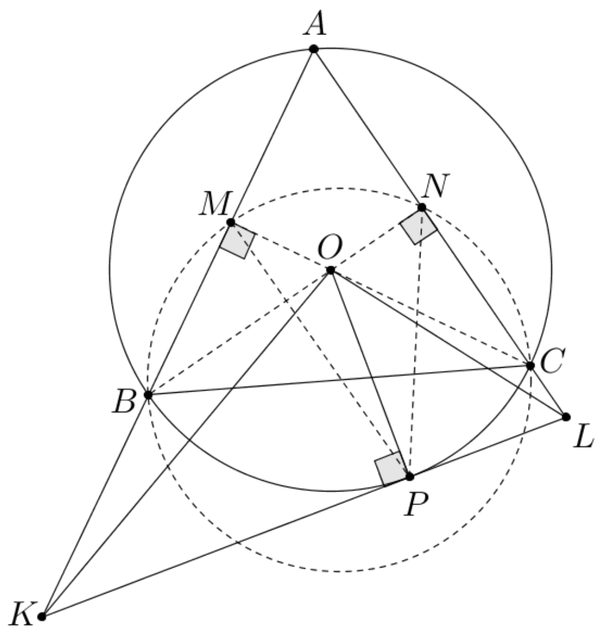
Solution 1.

Suppose that M and N be the midpoints of AB and AC respectively. We know that quadrilateral $BMNC$ is cyclic. Also $\angle BPC = 120° > 90°$, so we can say the point P is in the circumcircle of quadrilateral $BMNC$. Therefore: $\angle MPN > \angle MBN = 30°$

In the other hand, quadrilaterals $KMOP$ and $NOPL$ are cyclic. Therefore:

$$\angle MKO = \angle MPO \ , \ \angle NLO = \angle NPO \ \Rightarrow \ \angle AKO + \angle ALO = \angle MPN > 30°$$

$$\Rightarrow \angle KOL = \angle A + \angle AKO + \angle ALO > 90°$$



Solution 2.

Suppose that $\angle KOL \leq 90^\circ$, therfore $KL^2 \leq OK^2 + OL^2$. Assume that R is the radius of a circumcircle $\triangle ABC$. Let $BK = x$ and $LC = y$ and $AB = AC = BC = a$. According to law of cosines in triangle AKL, we have:

$$KL^2 = AK^2 + AL^2 - AK.AL.cos(\angle A) \quad \Rightarrow \quad KL^2 = (a+x)^2 + (a+y)^2 - (a+x)(a+y)$$

In the other hand:

$$KB.KA = OK^2 - R^2 \quad \Rightarrow \quad OK^2 = R^2 + x(a+x)$$

$$LC.LA = OL^2 - R^2 \quad \Rightarrow \quad OL^2 = R^2 + y(a+y)$$

We know that $KL^2 \leq OK^2 + OL^2$ and $a = R\sqrt{3}$, therfore:

$$(a+x)^2 + (a+y)^2 - (a+x)(a+y) \leq 2R^2 + x(a+x) + y(a+y)$$

$$\Rightarrow \quad R^2 \leq xy \quad (1)$$

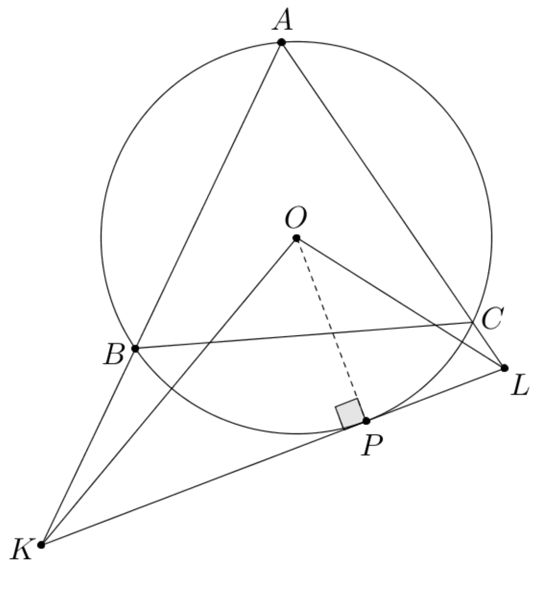
KL is tangent to circumcircle of $\triangle ABC$ at P. So we have:

$$KP^2 = KB.KA = x(a+x) > x^2 \quad \Rightarrow \quad KP > x \quad (2)$$

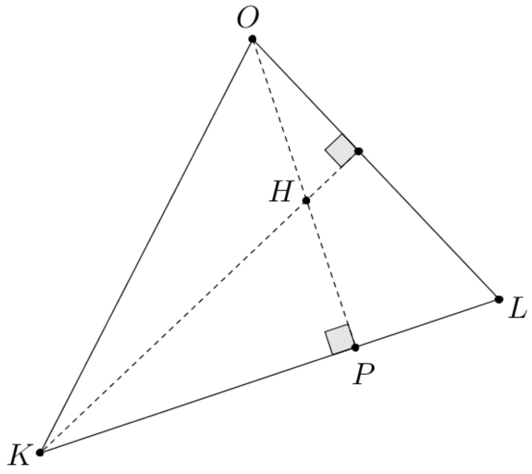
$$LP^2 = LC.LA = y(a+y) > y^2 \quad \Rightarrow \quad LP > y \quad (3)$$

According to inequality 2, 3 we can say: $\quad xy < KP.LP \quad (4)$

Now According to inequality 1, 4 we have: $\quad R^2 < KP.LP \quad (5)$



We know that $\angle KOL \leq 90°$, therefore KOL is acute-triangle. Suppose that H is orthocenter of $\triangle KOL$. So the point H lies on OP and we can say $HP \leq OP$.



In other hand, $\angle HKP = \angle POL$ and $\angle KHP = \angle OLP$, therefore two triangles THP and OPL are similar. So we have:

$$\frac{KP}{HP} = \frac{OP}{LP} \quad \Rightarrow \quad KP.LP = HP.OP \leq OP^2 = R^2$$

But according to inequality 5, we have $R^2 < KP.LP$ and it's a contradiction. Therfore $\angle KOL > 90°$.

5. a) Do there exist 5 circles in the plane such that every circle passes through centers of exactly 3 circles?

b) Do there exist 6 circles in the plane such that every circle passes through centers of exactly 3 circles?

Proposed by Morteza Saghafian

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a)Solution.

There aren't such 5 circles. Suppose that these circles exists, therefore their centers are 5 points that each point has same distance from 3 other points and has diffrent distance from the remaining point. We draw an arrow from each point to it's diffrent distance point.

- lemma 1. We don't have two points such O_i, O_j that each one is the diffrent distance point of the other one.

proof. If we have such thing then O_i and O_j both have same distance to the remaining points, therefore both of them are circumcenter of the remaining points, which is wrong.

- lemma 2. We don't have 4 points such O_i, O_j, O_k, O_l that O_i, O_j put their arrow in O_k and O_K puts it's arrow in O_l.

proof. If we name the remaining point O_m then the distances of O_i from O_j, O_l, O_m are equal and the distances of O_j from O_i, O_l, O_m are equal. Therefore each of O_l, O_m is the diffrent distance point of another which is wrong (according to lemma 1).

so each point sends an arrow and recives an arrow. Because of lemma 1 we don't have 3 or 4 points cycles. Therefore we only have one 5 points cycle. So each pair of these 5 points should have equal distance. which is impossible.

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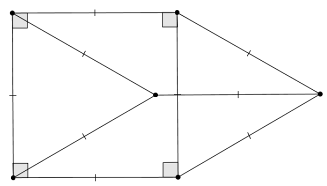
b)Solution.

in the picture below, we have 6 points in the plane such that for every point there exists exactly 3 other points on a circle with radius 1 centimeter.



Solutions of 2nd Iranian Geometry Olympiad 2015 (Advanced)

1. Two circles ω_1 and ω_2 (with centers O_1 and O_2 respectively) intersect at A and B. The point X lies on ω_2. Let point Y be a point on ω_1 such that $\angle XBY = 90^\circ$. Let X' be the second point of intersection of the line O_1X and ω_2 and K be the second point of intersection of $X'Y$ and ω_2. Prove that X is the midpoint of arc AK.

Proposed by Davood Vakili

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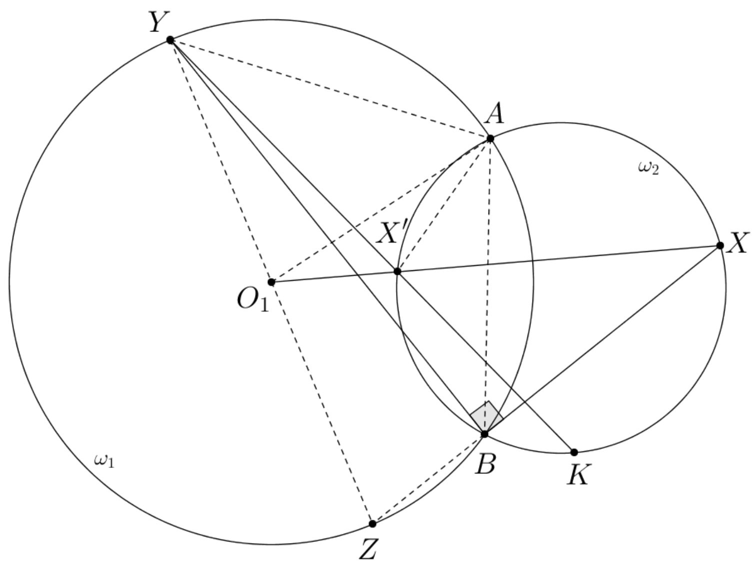
Solution.

Suppose that the point Z be the intersection of BX and circle ω_1. We know that $\angle YBZ = 90^\circ$, therefore the points Y, O_1, Z are collinear.

$$\angle O_1YA = \angle ABX = \angle AX'X \quad \Rightarrow \quad YAX'O_1 : cyclic$$

In the other hand, we know that $AO_1 = YO_1$ so $\angle AX'X = \angle YX'O_1 = \angle XX'K$. Therefore the point X lies on the midpoint of arc AK.



2. Let ABC be an equilateral triangle with circumcircle ω and circumcenter O. Let P be the point on the arc BC(the arc which A doesn't lie). Tangent to ω at P intersects extensions of AB and AC at K and L respectively. Show that $\angle KOL > 90°$.

Proposed by Iman Maghsoudi

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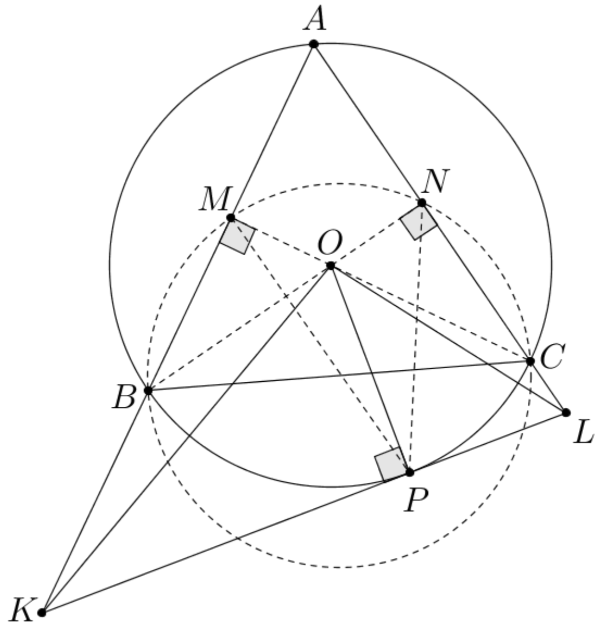
Solution 1.

Suppose that M and N be the midpoints of AB and AC respectively. We know that quadrilateral $BMNC$ is cyclic. Also $\angle BPC = 120° > 90°$, so we can say the point P is in the circumcircle of quadrilateral $BMNC$. Therefore: $\angle MPN > \angle MBN = 30°$

In the other hand, quadrilaterals $KMOP$ and $NOPL$ are cyclic. Therefore:

$$\angle MKO = \angle MPO \ , \ \angle NLO = \angle NPO \ \Rightarrow \ \angle AKO + \angle ALO = \angle MPN > 30°$$

$$\Rightarrow \angle KOL = \angle A + \angle AKO + \angle ALO > 90°$$



Solution 2.

Suppose that $\angle KOL \leq 90^\circ$, therfore $KL^2 \leq OK^2 + OL^2$. Assume that R is the radius of a circumcircle $\triangle ABC$. Let $BK = x$ and $LC = y$ and $AB = AC = BC = a$. According to law of cosines in triangle AKL, we have:

$$KL^2 = AK^2 + AL^2 - AK.AL.cos(\angle A) \quad \Rightarrow \quad KL^2 = (a+x)^2 + (a+y)^2 - (a+x)(a+y)$$

In the other hand:

$$KB.KA = OK^2 - R^2 \quad \Rightarrow \quad OK^2 = R^2 + x(a+x)$$

$$LC.LA = OL^2 - R^2 \quad \Rightarrow \quad OL^2 = R^2 + y(a+y)$$

We know that $KL^2 \leq OK^2 + OL^2$ and $a = R\sqrt{3}$, therfore:

$$(a+x)^2 + (a+y)^2 - (a+x)(a+y) \leq 2R^2 + x(a+x) + y(a+y)$$

$$\Rightarrow \quad R^2 \leq xy \quad (1)$$

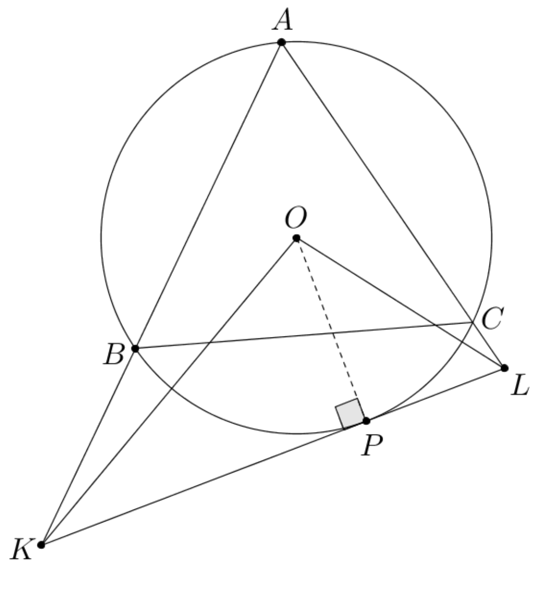
KL is tangent to circumcircle of $\triangle ABC$ at P. So we have:

$$KP^2 = KB.KA = x(a+x) > x^2 \quad \Rightarrow \quad KP > x \quad (2)$$

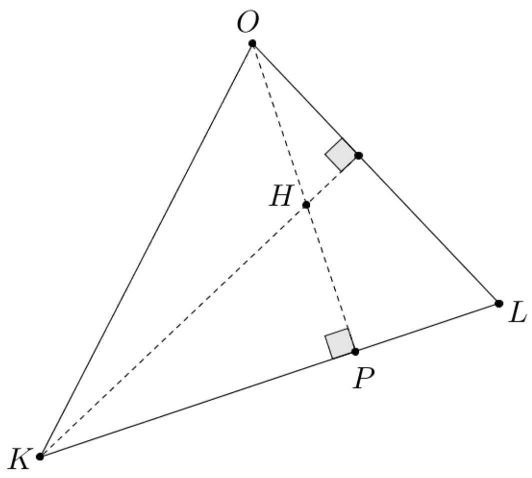
$$LP^2 = LC.LA = y(a+y) > y^2 \quad \Rightarrow \quad LP > y \quad (3)$$

According to inequality 2, 3 we can say: $\quad xy < KP.LP \quad (4)$

Now According to inequality 1, 4 we have: $\quad R^2 < KP.LP \quad (5)$



We know that $\angle KOL \leq 90°$, therefore KOL is acute-triangle. Suppose that H is orthocenter of $\triangle KOL$. So the point H lies on OP and we can say $HP \leq OP$.



In other hand, $\angle HKP = \angle POL$ and $\angle KHP = \angle OLP$, therefore two triangles THP and OPL are similar. So we have:

$$\frac{KP}{HP} = \frac{OP}{LP} \quad \Rightarrow \quad KP.LP = HP.OP \leq OP^2 = R^2$$

But according to inequality 5, we have $R^2 < KP.LP$ and it's a contradiction. Therfore $\angle KOL > 90°$.

3. Let H be the orthocenter of the triangle ABC. Let l_1 and l_2 be two lines passing through H and perpendicular to each other. l_1 intersects BC and extension of AB at D and Z respectively, and l_2 intersects BC and extension of AC at E and X respectively. Let Y be a point such that $YD \parallel AC$ and $YE \parallel AB$. Prove that X, Y, Z are collinear.

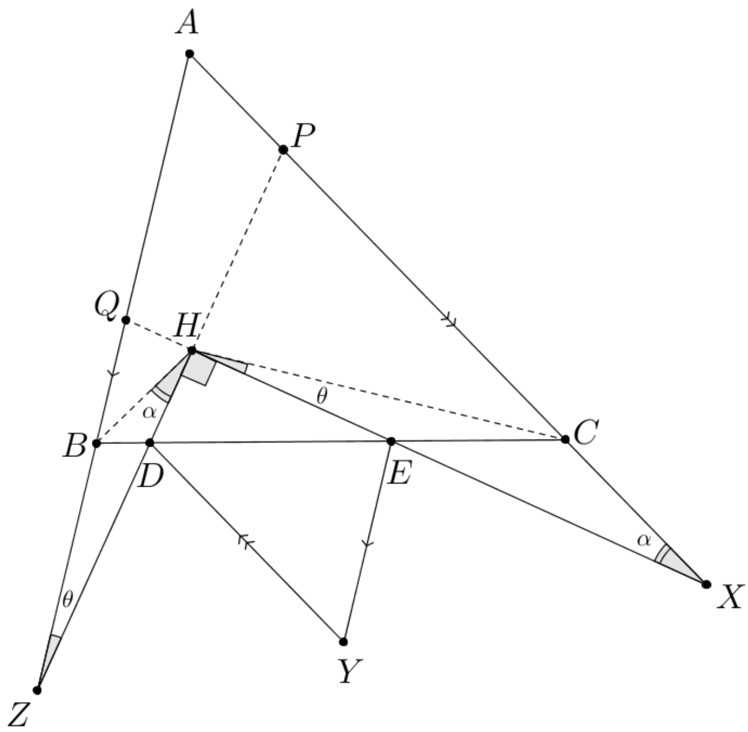
Proposed by Ali Golmakani

Solution.

Suppose that HZ intersects AC at P and HX intersects AB at Q. According to Menelaus's theorem in two triangles AQX and APZ we can say:

$$\frac{CX}{AC} \cdot \frac{AB}{BQ} \cdot \frac{QE}{EX} = 1 \qquad (1) \qquad \textit{and} \qquad \frac{BZ}{AB} \cdot \frac{AC}{PC} \cdot \frac{PD}{DZ} = 1 \qquad (2)$$

In the other hand, H is the orthocenter of $\triangle ABC$. So $BH \perp AC$ and we know that $\angle DHE = 90°$, therefore $\angle HXA = \angle BHZ = \alpha$. Similarly we can say $\angle HZA = \angle CHX = \theta$.



According to law of sines in $\triangle HPC$, $\triangle HCX$ and $\triangle HPX$:

$$\frac{sin(90-\theta)}{PC}=\frac{sin(\angle HCP)}{HP} \quad , \quad \frac{sin(\theta)}{CX}=\frac{sin(\angle HCX)}{HX} \quad , \quad \frac{HP}{HX}=\frac{sin(\alpha)}{sin(90-\alpha)}$$

$$\Rightarrow \quad \frac{PC}{CX}=\frac{tan(\alpha)}{tan(\theta)}$$

Similarly, according to law of sines in $\triangle HBQ$, $\triangle HBZ$ and $\triangle HQZ$, we can show:

$$\Rightarrow \quad \frac{BZ}{BQ}=\frac{tan(\alpha)}{tan(\theta)} \quad \Rightarrow \quad \frac{BZ}{BQ}=\frac{PC}{CX} \quad \Rightarrow \quad \frac{PC}{BZ}=\frac{CX}{BQ} \qquad (3)$$

According to equality 1, 2 and 3, we can say:

$$\frac{XE}{EQ}=\frac{PD}{ZD} \qquad (4)$$

Suppose that the line which passes through E and parallel to AB, intersects ZX at Y_1 and the line which passes through D and parallel to AC, intersects ZX at Y_2. According to Thales's theorem we can say:

$$\frac{Y_1X}{ZY_1}=\frac{XE}{EQ} \quad , \quad \frac{Y_2X}{ZY_2}=\frac{PD}{ZD}$$

According to equality 4, we show that $Y_1 \equiv Y_2$, therefore the point Y lies on ZX.

4. In triangle ABC, we draw the circle with center A and radius AB. This circle intersects AC at two points. Also we draw the circle with center A and radius AC and this circle intersects AB at two points. Denote these four points by A_1, A_2, A_3, A_4. Find the points B_1, B_2, B_3, B_4 and C_1, C_2, C_3, C_4 similarly. Suppose that these 12 points lie on two circles. Prove that the triangle ABC is isosceles.

Proposed by Morteza Saghafian

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Solution 1.

Suppose that triangle ABC isn't isosceles and $a > b > c$. In this case, there are four points (from these 12 points) on each side of $\triangle ABC$. Suppose that these 12 points lie on two circles ω_1 and ω_2. Therefore each one of the circles ω_1 and ω_2 intersects each side of $\triangle ABC$ exactly at two points. Suppose that $P(A,\omega_1), P(A,\omega_2)$ are power of the point A with respect to circles ω_1, ω_2 respectively. Now we know that:

$$P(A,\omega_1).P(A,\omega_2) = b.b.(a-c).(a+c) = c.c.(a-b)(a+b)$$

$$\Rightarrow \quad b^2(a^2-c^2) = c^2(a^2-b^2) \quad \Rightarrow \quad a^2(b^2-c^2) = 0 \quad \Rightarrow \quad b = c$$

But we know that $b > c$ and it's a contradiction. Therefore the triangle ABC is isosceles.

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Solution 2.

Suppose that triangle ABC isn't isosceles. In this case, there are four points (from these 12 points) on each side of $\triangle ABC$. Suppose that these 12 points lie on two circles ω_1 and ω_2. Therefore each one of the circles ω_1 and ω_2 intersects each side of $\triangle ABC$ exactly at two points (and each one of the circles ω_1 and ω_2 doesn't pass through A, B, C). We know that the intersections of ω_1 and the sides of $\triangle ABC$ is even number. Also the intersections of ω_2 and the sides of $\triangle ABC$ is even number. But Among the these 12 points, just 3 points lie on the sides of $\triangle ABC$ and this is odd number. So it's a contradiction. Therefore the triangle ABC is isosceles.

5. Rectangles ABA_1B_2, BCB_1C_2, CAC_1A_2 lie otside triangle ABC. Let C' be a point such that $C'A_1 \perp A_1C_2$ and $C'B_2 \perp B_2C_1$. Points A' and B' are defined similarly. Prove that lines AA', BB', CC' concur.

Proposed by Alexey Zaslavsky (Russia)

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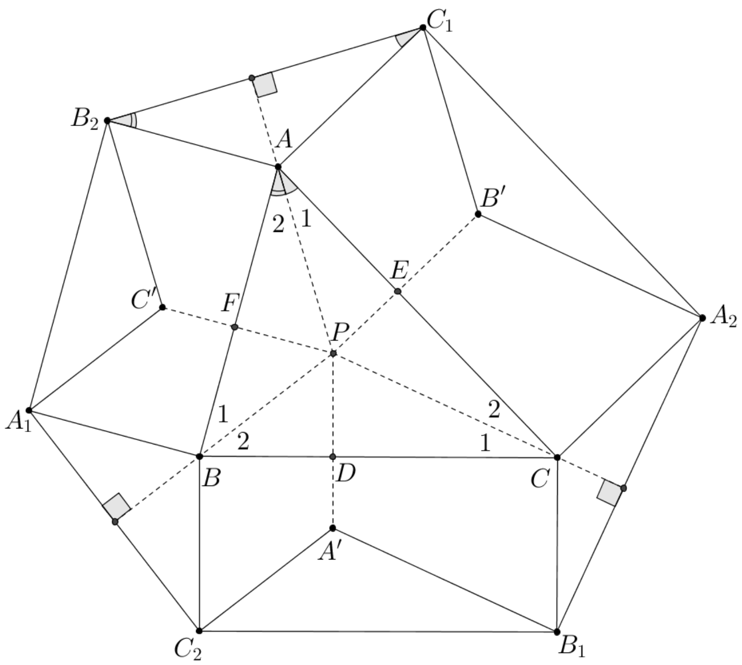
Solution.

Suppose that l_A is the line which passes through A and perpendicular to B_2C_1. Let l_B and l_C similarly. Suppose that $CB_1 = BC_2 = x$ and $BA_1 = AB_2 = y$ and $AC_1 = CA_2 = z$. According to angles equality, we can say:

$$\frac{sin(\angle A_1)}{sin(\angle A_2)} = \frac{y}{z} \quad , \quad \frac{sin(\angle B_1)}{sin(\angle B_2)} = \frac{x}{y} \quad , \quad \frac{sin(\angle C_1)}{sin(\angle C_2)} = \frac{z}{x}$$

According to sine form of Ceva's theorem in $\triangle ABC$, l_A, l_B, l_C are concur. Suppose that l_A, l_B, l_C pass through the point P. We know that $\triangle PBC$ and $\triangle A'C_2B_1$ are equal. (because of $BP \parallel A'C_2$, $CP \parallel A'B_1$, $BC \parallel B_1C_2$ and $BC = B_1C_2$). So we have:

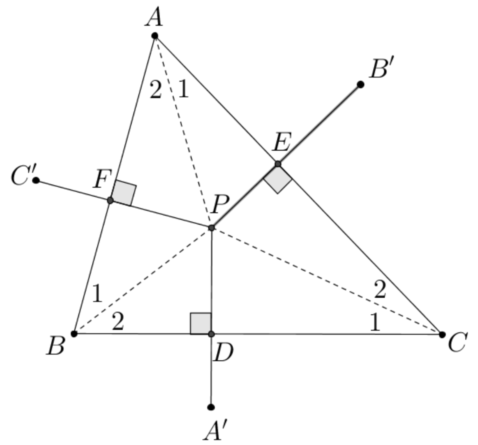
$$PA' = x \ , \ PC' = y \ , \ PB' = z \qquad PA' \perp BC \ , \ PB' \perp AC \ , \ PC' \perp AB$$



Suppose that PA', PB', PC' intersects BC, AC, AB at D, E, F respectively and: $PD = m$, $PE = n$, $PF = t$. According to before figure, we have:

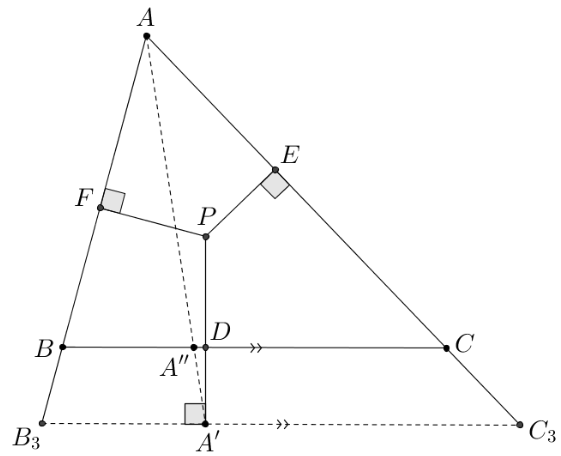
$$\frac{sin(\angle A_1)}{sin(\angle A_2)} = \frac{n}{t} = \frac{y}{z} \quad , \quad \frac{sin(\angle B_1)}{sin(\angle B_2)} = \frac{t}{m} = \frac{x}{y} \quad , \quad \frac{sin(\angle C_1)}{sin(\angle C_2)} = \frac{m}{n} = \frac{z}{x}$$

If $n = ky$, then: $t = kz$, $m = \frac{kyz}{x}$.



Now draw the line from A' such that be parallel to BC. The intersection of this line and extension AB and AC denote by B_3 and C_3 respectively. Let the point A'' be the intersection of AA' and BC. According to Thales's theorem, we have:

$$\frac{BA''}{CA''} = \frac{B_3A'}{C_3A'}$$



Let $\angle B_3PA' = \alpha$ and $\angle C_3PA' = \theta$. We know that the quadrilaterals PFB_3A' and PEC_3A' are cyclic. Therefore $\angle B_3FA' = \alpha$ and $\angle C_3EA' = \theta$.

According to law of sines in $\triangle PB_3A'$ and $\triangle PC_3A'$ and $\triangle PC_3B_3$:

$$\frac{B_3A'}{C_3A'} = \frac{tan(\alpha)}{tan(\theta)}$$

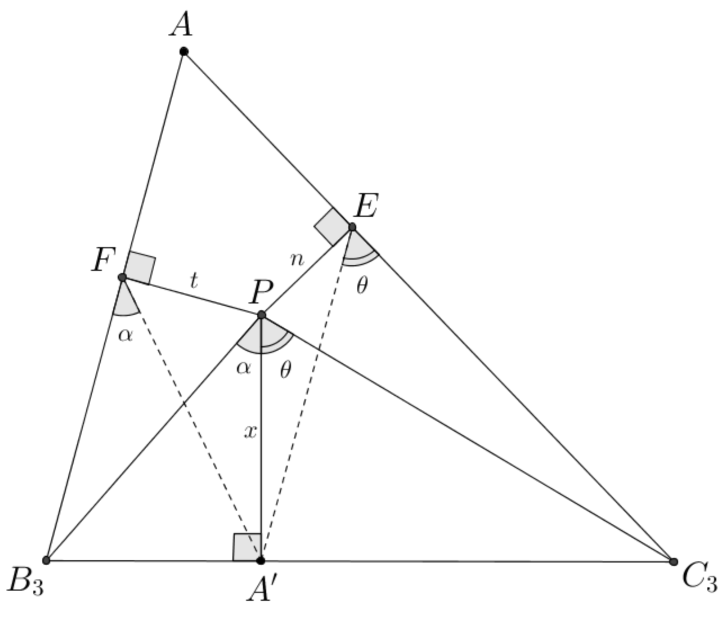
Also according to law of sines in $\triangle PFA'$:

$$\frac{t}{x} = \frac{sin(\angle B + \alpha - 90)}{cos(\alpha)} = \frac{cos(\angle B + \alpha)}{cos(\alpha)} = cos(\angle B) - tan(\alpha).sin(\angle B)$$

$$\Rightarrow \quad tan(\alpha) = \frac{cos(\angle B) - \frac{t}{x}}{sin(\angle B)}$$

Similarly we can say:

$$tan(\theta) = \frac{cos(\angle C) - \frac{n}{x}}{sin(\angle C)} \quad \Rightarrow \quad \frac{B_3A'}{C_3A'} = \frac{BA''}{CA''} = \frac{x.cos(\angle B) - t}{x.cos(\angle C) - n}.\frac{sin(\angle C)}{sin(\angle B)}$$



Similarly, two other fractions can be calculated.

According to Ceva's theorem in $\triangle ABC$, we have to that:

$$\frac{x.cos(\angle B)-t}{x.cos(\angle C)-n}.\frac{sin(\angle C)}{sin(\angle B)}.\frac{z.cos(\angle C)-m}{z.cos(\angle A)-t}.\frac{sin(\angle A)}{sin(\angle C)}.\frac{y.cos(\angle A)-n}{y.cos(\angle B)-m}.\frac{sin(\angle B)}{sin(\angle A)}=1$$

$$\Longleftrightarrow \frac{x.cos(\angle B)-t}{x.cos(\angle C)-n}.\frac{z.cos(\angle C)-m}{z.cos(\angle A)-t}.\frac{y.cos(\angle A)-n}{y.cos(\angle B)-m}=1$$

In other hand, we know that:

$$n=ky \quad , \quad t=kz \quad , \quad m=\frac{kyz}{x}$$

$$\Longleftrightarrow \frac{x.cos(\angle B)-kz}{x.cos(\angle C)-ky}.\frac{x.cos(\angle C)-ky}{x.cos(\angle A)-kx}.\frac{x.cos(\angle A)-kx}{x.cos(\angle B)-kz}=1$$

Therfore, we show that AA', BB', CC' are concur.